Hölder Functionals and Quotients

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Abstract

We describe an inequality of finite or infinite sequences of real numbers and their quotients. More precisely, we compare the quotient of Hölder functionals of two sequences of numbers with the sum of their quotients. In the last section we investigate the 'wideness' of the inequality, i.e. we show that both the inequality can converge into an equality, and the difference between the two sides of the inequality can be arbitrary large.

1 Introduction

For convenience, we restrict our considerations on positive real numbers, i.e.

$$a_1, a_2, a_3, \ \dots \quad b_1, b_2, b_3, \ \dots \quad > 0 \ ,$$

to avoid problems with a denominator 0, otherwise we have to discuss cases with expressions like '$\frac{0}{0}$' and '$\frac{a}{0}$'. Further, it is easy to extend the coming theorems on negative numbers by using the modulus of a number.

We start with a known statement wich is called the 'Rearrangement Inequality'.

Theorem 1. *Let us take two finite ordered sequences of positive real numbers of the same length, i.e. we have*

$$0 < a_1 \leq a_2 \leq a_3 \leq \ \dots \ \leq a_n \quad \textit{and} \quad 0 < b_1 \leq b_2 \leq b_3 \leq \ \dots \ \leq b_n$$

for a natural number n. Let σ be any permutation on the set $\{1, 2, 3, \dots, n\}$. We have the inequality of sums of fractions

$$\sum_{k=1}^{n} \frac{a_k}{b_k} \ \leq \ \sum_{k=1}^{n} \frac{a_k}{b_{\sigma(k)}} \ \leq \ \sum_{k=1}^{n} \frac{a_k}{b_{n-k+1}} \ .$$

Proof. We can find it in [1]. □

We say $\vec{a}$ for any n-tuple $(a_1, a_2, a_3, \dots, a_n)$ of positive real numbers.
We fix any non-zero real number p. Note that the following expression

$$\|\vec{a}\|_p := \sqrt[p]{a_1^p + \ a_2^p + \ a_3^p + \ \dots \ + \ a_n^p}$$

is defined for all $p \neq 0$, since all a_k are positive. We call $\|\vec{a}\|_p$ a *Hölder functional* of the tuple $\vec{a} = (a_1, a_2, \dots, a_n)$.

Now we are prepared for the main theorem.

Theorem 2 (Main inequality). *Let us take Hölder functionals of two finite sequences of positive real numbers of the same length* $n \in \mathbb{N}$, $n \geq 2$. *That means we have* $0 < a_1, a_2, a_3, \ \ldots \ , \ a_n$ *and* $0 < b_1, b_2, b_3, \ \ldots \ , \ b_n$. *For all real numbers* $p \neq 0$ *there is the following strict inequality*

$$\frac{\|\vec{a}\|_p}{\|\vec{b}\|_p} \ = \ \sqrt[p]{\frac{a_1^p + \ a_2^p + \ a_3^p + \ \ldots \ + \ a_n^p}{b_1^p + \ b_2^p + \ b_3^p + \ \ldots \ + \ b_n^p}} \ < \ \sum_{k=1}^{n} \frac{a_k}{b_k} \ .$$

The inequality remains valid also for the limits $p = -\infty$, $p = \infty$, *and* $p = 0$. *Further, the inequality is sharp, i.e. in this generality it can not be improved. Further, for* $n = 1$ *we have a trivial equality.*

Note that the arrangement of the numbers $a_1, a_2, a_3, \ \ldots \ , \ a_n$ and $b_1, b_2, b_3, \ \ldots \ , \ b_n$ does not affect the left hand side of the inequality but the right hand side. If the inequality is true, it must be true in the 'worst' case, i.e. for the arrangement

$$0 < a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \quad \text{and} \quad 0 < b_1 \leq b_2 \leq b_3 \leq \ldots \leq b_n \ ,$$

see Theorem (1). By this arrangement the right hand side is as small as possible.

In the next section we prove the theorem, which is the main contribution of this article. In the last section we show that the difference between both sides of the inequality can become arbitrary small and arbitrary big.

2 The Proof of the Inequality

Proof. The way of proving the theorem is not surprising. We prove it for $n = 2$, which is the harder part, and after that we go the induction step from n to $n + 1$.

Beginning of the induction: Let $n = 2$. Let us take positive numbers a_1, a_2, b_1, b_2 with the arrangement $0 < a_1 \leq a_2$ and $0 < b_1 \leq b_2$. We want to prove that

$$\sqrt[p]{\frac{a_1^p + \ a_2^p}{b_1^p + \ b_2^p}} \ < \ \frac{a_1}{b_1} + \frac{a_2}{b_2} \tag{1}$$

holds for all real numbers $p \neq 0$. Since $a_1 \leq a_2$ and $b_1 \leq b_2$ there are two positive numbers $0 < \alpha, \beta \leq 1$ with $a_1 = \alpha \cdot a_2$ and $b_1 = \beta \cdot b_2$. Inequality (1) is equivalent to

$$\sqrt[p]{\frac{\alpha^p + 1}{\beta^p + 1}} \ < \ \frac{\alpha}{\beta} + \frac{1}{1} \ = \ \frac{\alpha}{\beta} + 1 \ . \tag{2}$$

We distinguish four cases. The Case **D** deals with a negative p.

- Case **A**: $0 < p$ and $\alpha \leq \beta$,
- Case **B**: $0 < p \leq 1$ and $\beta < \alpha$,
- Case **C**: $1 < p$ and $\beta < \alpha$,
- Case **D**: $p < 0$.

Case **A**: We assume $0 < p$ and $\alpha \leq \beta$. We have

$$\sqrt[p]{\frac{\alpha^p + 1}{\beta^p + 1}} \ \leq \ \sqrt[p]{\frac{\alpha^p + 1}{\alpha^p + 1}} \ = \ 1 \ < \ \frac{\alpha}{\beta} + 1 \ . \tag{3}$$

Case **B**: We assume $0 < p \le 1$ and $\beta < \alpha$. We set $q := \frac{1}{p}$, hence $1 \le q$. We want to prove Inequality (2), we write it again as

$$\sqrt[p]{\frac{\alpha^p + 1}{\beta^p + 1}} \;=\; \left[\frac{\sqrt[q]{\alpha} + 1}{\sqrt[q]{\beta} + 1}\right]^q \;<\; \frac{\alpha}{\beta} + 1\ . \tag{4}$$

We have the following chain of equivalences to the desired Inequality (2)

$$\begin{aligned}
\left[\frac{\sqrt[q]{\alpha} + 1}{\sqrt[q]{\beta} + 1}\right]^q \;<\; \frac{\alpha}{\beta} + 1 \;&\Longleftrightarrow\; \left[\sqrt[q]{\alpha} + 1\right]^q \;<\; \left[\sqrt[q]{\beta} + 1\right]^q \cdot \left[\frac{\alpha}{\beta} + 1\right] \\
&\Longleftrightarrow\; \sqrt[q]{\alpha} + 1 \;<\; \left[\sqrt[q]{\beta} + 1\right] \cdot \sqrt[q]{\frac{\alpha}{\beta} + 1} \\
&\Longleftrightarrow\; \sqrt[q]{\alpha} + 1 \;<\; \sqrt[q]{\alpha + \beta} \;+\; \sqrt[q]{\frac{\alpha}{\beta} + 1}\ .
\end{aligned}$$

The last inequality is obvious, which finishes Case **B**.
Case **C**: We assume $1 < p$ and $\beta < \alpha$. We can write the chain of inequalities

$$\sqrt[p]{\frac{\alpha^p + 1}{\beta^p + 1}} \;<\; \sqrt[p]{\frac{\alpha^p + 1}{1}} \;\le\; \sqrt[p]{2} \;<\; 2 \;<\; \frac{\alpha}{\beta} + 1\,,$$

and Case **C** is proven. Therefore Inequality (1) is shown for all real $p > 0$.
Case **D**: We investigate the case of a negative real number p, i.e. let $p < 0$. We define $q := -p$, i.e. q is a positive number, hence we can refer to the first three cases. We write

$$\sqrt[p]{\frac{\alpha^p + 1}{\beta^p + 1}} \;=\; \sqrt[-q]{\frac{\alpha^{-q} + 1}{\beta^{-q} + 1}} \;=\; \sqrt[q]{\frac{\left[\frac{1}{\beta}\right]^q + 1}{\left[\frac{1}{\alpha}\right]^q + 1}} \;<\; \frac{\frac{1}{\beta}}{\frac{1}{\alpha}} + \frac{1}{1} \;=\; \frac{\alpha}{\beta} + 1\ .$$

This shows Case **D**, and the last of four cases to prove the beginning of the induction with $n = 2$ is done.

Induction step: Let the theorem be proven for a natural number $n \ge 2$. We prove it for the next number $n + 1$. Let us take two sets of $n + 1$ positive numbers, i.e. we assume $a_1, a_2, a_3, \ \ldots\ , a_n, a_{n+1}$ and $b_1, b_2, b_3, \ \ldots\ , b_n, b_{n+1} \;>\; 0$. We have

$$\begin{aligned}
\sqrt[p]{\frac{a_1^p + a_2^p + \ \ldots\ + a_n^p + a_{n+1}^p}{b_1^p + b_2^p + \ \ldots\ + b_n^p + b_{n+1}^p}} \;&=\; \sqrt[p]{\frac{\left(\sum_{k=1}^{n-1} a_k^p\right) \;+\; a_n^p \;+\; a_{n+1}^p}{\left(\sum_{k=1}^{n-1} b_k^p\right) \;+\; b_n^p \;+\; b_{n+1}^p}} \\
&=\; \sqrt[p]{\frac{\left(\sum_{k=1}^{n-1} a_k^p\right) \;+\; \left[\sqrt[p]{a_n^p \;+\; a_{n+1}^p}\right]^p}{\left(\sum_{k=1}^{n-1} b_k^p\right) \;+\; \left[\sqrt[p]{b_n^p \;+\; b_{n+1}^p}\right]^p}} \\
&<\; \left(\sum_{k=1}^{n-1} \frac{a_k}{b_k}\right) \;+\; \frac{\sqrt[p]{a_n^p \;+\; a_{n+1}^p}}{\sqrt[p]{b_n^p \;+\; b_{n+1}^p}} \\
&<\; \left(\sum_{k=1}^{n-1} \frac{a_k}{b_k}\right) \;+\; \frac{a_n}{b_n} \;+\; \frac{a_{n+1}}{b_{n+1}}\ .
\end{aligned}$$

This was the induction step $n \to n + 1$, and the inequality of Theorem (2) is proven.

To complete the proof we have to consider the cases $p = \infty$, $p = -\infty$ and $p = 0$. Because the inequality is proven for real numbers $p \ne 0$, it should be valid also for the limits

$p = \infty$, $p = -\infty$ and $p = 0$. But we prefer to compute these three cases. Finally, we say something about the statement that 'the inequality is sharp'.

For an n-tuple $\vec{a} = (a_1, a_2, \ \ldots, \ a_n) \in \mathbb{R}^n$ of positive numbers let

$$A := \max\{a_1, a_2, \ \ldots \ , \ a_n\} \quad \text{and} \quad a := \min\{a_1, a_2, \ \ldots \ , \ a_n\}.$$

The following limits are well known and easy to proof.

$$\lim_{p \to +\infty} (\|\vec{a}\|_p) = A \ , \quad \text{and} \quad \lim_{p \to -\infty} (\|\vec{a}\|_p) = a \ .$$

For n-tuples $\vec{a}$ and $\vec{b}$ (without restriction of generality) we choose the arrangements

$$0 < a = a_1 \le a_2 \le a_3 \le \ \ldots \ \le a_n = A \quad \text{and} \quad 0 < b := b_1 \le b_2 \le b_3 \le \ \ldots \ \le b_n =: B.$$

It follows very easily that the limits are

$$\lim_{p \to +\infty} \left(\frac{\|\vec{a}\|_p}{\|\vec{b}\|_p} \right) = \frac{A}{B} < \sum_{k=1}^{n} \frac{a_k}{b_k} \quad \text{and} \quad \lim_{p \to -\infty} \left(\frac{\|\vec{a}\|_p}{\|\vec{b}\|_p} \right) = \frac{a}{b} < \sum_{k=1}^{n} \frac{a_k}{b_k} \ .$$

The case $p = 0$ needs more attention.

We abbreviate the left hand side of the inequality in Theorem (2) by

$$\mathsf{L}HS_p := \frac{\|\vec{a}\|_p}{\|\vec{b}\|_p} = \frac{\sqrt[p]{a_1^p + \ a_2^p + \ a_3^p + \ \ldots \ + \ a_n^p}}{\sqrt[p]{b_1^p + \ b_2^p + \ b_3^p + \ \ldots \ + \ b_n^p}}$$

for arbitrary n-tuples $\vec{a}$ and $\vec{b}$ of positive numbers. Instead of $\mathsf{L}HS_p$ we consider $\log(\mathsf{L}HS_p)$. If we set $p := 0$ in $\log(\mathsf{L}HS_p)$ we would have a numerator and a denominator 0, hence we can use the rules of L'Hospital. We compute the limit $\log(\mathsf{L}HS_p)$ for $p \to 0$, and we calculate with some effort

$$\lim_{p \to 0} (\log[\mathsf{L}HS_p]) = \lim_{p \to 0} \left(\frac{1}{p} \cdot \log \left[\frac{a_1^p + \ a_2^p + \ \ldots \ + \ a_n^p}{b_1^p + \ b_2^p + \ \ldots \ + \ b_n^p} \right] \right) = \frac{1}{n} \cdot \sum_{k=1}^{n} \log(a_k) - \log(b_k) \ .$$

We have $\lim_{p \to 0} [\mathsf{L}HS_p] = \lim_{p \to 0} [\exp(\log(\mathsf{L}HS_p))] = \exp\left(\lim_{p \to 0} [\log(\mathsf{L}HS_p)]\right)$. Hence we get the limit

$$\lim_{p \to 0} \left(\frac{\|\vec{a}\|_p}{\|\vec{b}\|_p} \right) = \lim_{p \to 0} [\mathsf{L}HS_p] = \exp\left(\frac{1}{n} \cdot \sum_{k=1}^{n} \log(a_k) - \log(b_k) \right) = \sqrt[n]{\prod_{k=1}^{n} \frac{a_k}{b_k}} \ .$$

Since there is the well-known inequality between the geometric and the arithmetic mean we finally get the desired inequality

$$\lim_{p \to 0} \left(\frac{\|\vec{a}\|_p}{\|\vec{b}\|_p} \right) = \sqrt[n]{\prod_{k=1}^{n} \frac{a_k}{b_k}} \le \frac{1}{n} \cdot \left(\sum_{k=1}^{n} \frac{a_k}{b_k} \right) < \left(\sum_{k=1}^{n} \frac{a_k}{b_k} \right) , \tag{5}$$

and the case $p = 0$ of our main Theorem (2) is proven.

The last point we have to discuss is the remark in Theorem (2) that 'in this generality it can not be improved'. It means that there are special cases such that instead of an inequation we almost have an equation.

Let p be any positve number, and let

$$b_1 = b_2 = b_3 = \ \ldots \ = b_n := 1, \ \text{and also} \ a_n := 1.$$

Further define $a_1 = a_2 = \ \ldots \ = a_{n-1} := \frac{1}{p}$, and the tuples $\vec{a} := (a_1, a_2, \ldots, a_n)$ and $\vec{b} :=$

$(b_1, b_2, \dots, b_n)$.
We have just provided the proof of the inequality

$$\frac{\|\vec{a}\|_p}{\|\vec{b}\|_p} \;<\; \sum_{k=1}^{n} \frac{a_k}{b_k} \;=\; \left(\sum_{k=1}^{n-1} \frac{1}{p} \right) + \frac{1}{1} \;=\; \frac{n-1}{p} + 1 \,. \tag{6}$$

For all $p \geq 1$ we have $\max\{a_1, a_2, \dots, a_n\} = 1 = \max\{b_1, b_2, \dots, b_n\}$, and we get the limits

$$\lim_{p \to +\infty} \left(\|\vec{a}\|_p \right) = 1 \;\text{ and }\; \lim_{p \to +\infty} \left(\|\vec{b}\|_p \right) = 1 \,.$$

Hence both the left hand side and the right hand side of Inequality (6) converge to the number 1 if p converges to infinity. It means that the inequality converges into an equality. Therefore the inequality in our main theorem (2) can not be improved by insertion of a constant factor less than 1.
Finally, the last point of Theorem (2) has been discussed and the proof is finished. □

We formulate Theorem (2) for infinite sequences.

Corollary 1. Let us assume two infinite sequences (convergent or not)
$a_1, a_2, a_3, \ \dots, \ a_n, a_{n+1}, a_{n+2}, \ \dots\dots$ and $b_1, b_2, b_3, \ \dots, \ b_n, b_{n+1}, b_{n+2}, \ \dots\dots$
of positive numbers. We have the following inequality for all real numbers $p \neq 0$

$$a_1^p + a_2^p + a_3^p + \ \dots \ + a_n^p + \ \dots\dots \quad \leq \quad (b_1^p + b_2^p + b_3^p + \ \dots \ + \ b_n^p + \ \dots\dots) \cdot \left(\sum_{k=1}^{\infty} \frac{a_k}{b_k} \right)^p \,.$$

The corollary means among other things that if the left hand side converges to infinity, the right hand side must do the same.

3 Examples

In the remark after Inequality (6) we showed that the inequality of our Theorem (2) can converge into an equality. Here we add a further example. A third example shows that the difference between both sides of the main inequality can become arbitrary big.

Let n be a natural number, $n \geq 2$. We define n-tuples of positive numbers for each natural number K, let $\vec{a}_K := (a_1, a_2, \dots, a_n)$ and $\vec{b}_K := (b_1, b_2, \dots, b_n)$. and we define for every number $K \in \mathbb{N}$
$a_1 = a_2 = \ \dots \ = a_{n-1} := 10^{-(2 \cdot K)}$, and $b_1 = b_2 = \ \dots \ = b_{n-1} := 10^{-K}$,
and let $a_n = b_n := 1$. With our main inequality we get for each $p \neq 0$

$$\sqrt[p]{\frac{(n-1) \cdot 10^{-(p \cdot 2 \cdot K)} + 1}{(n-1) \cdot 10^{-(p \cdot K)} + 1}} \;=\; \sqrt[p]{\frac{a_1^p + \ a_2^p + \ \dots \ + \ a_n^p}{b_1^p + \ b_2^p + \ \dots \ + \ b_n^p}} \;<\; \sum_{k=1}^{n} \frac{a_k}{b_k} \;=\; \frac{n-1}{10^K} + 1 \,.$$

We fix any $p > 0$. If K converges to infinity we get the case that both sides of the inequality converge to the constant 1.

We change the parts of $\vec{a}_K$ and $\vec{b}_K$, and we define
$a_1 = a_2 = \ \dots \ = a_{n-1} := 10^{-K}$, and $b_1 = b_2 = \ \dots \ = b_{n-1} := 10^{-(2 \cdot K)}$,
and let $a_n = b_n := 1$ as before. In this case we get for each $p \neq 0$

$$\sqrt[p]{\frac{(n-1) \cdot 10^{-(p \cdot K)} + 1}{(n-1) \cdot 10^{-(p \cdot 2 \cdot K)} + 1}} \;=\; \sqrt[p]{\frac{a_1^p + \ a_2^p + \ \dots \ + \ a_n^p}{b_1^p + \ b_2^p + \ \dots \ + \ b_n^p}} \;<\; \sum_{k=1}^{n} \frac{a_k}{b_k} \;=\; (n-1) \cdot 10^K \ + \ 1 \,.$$

We fix $p > 0$. If K converges to infinity we see that the left hand side converges to the constant 1, while the right hand side converges to infinity.

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